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2007 J. Phys. A: Math. Theor. 40 14403

(http://iopscience.iop.org/1751-8121/40/48/007)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 14403–14412 doi:10.1088/1751-8113/40/48/007

Peculiarities of the hidden nonlinear supersymmetry of the Pöschl–Teller system in the light of the Lamé equation

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Received 29 August 2007, in final form 18 October 2007
Published 14 November 2007
Online at stacks.iop.org/JPhysA/40/14403

Abstract

A hidden nonlinear bosonized supersymmetry was revealed recently in the Pöschl–Teller and finite-gap Lamé systems. In spite of the intimate relationship between the two quantum models, the hidden supersymmetry in them displays essential differences. In particular, the kernel of the supercharges of the Pöschl–Teller system, unlike the case of the Lamé equation, includes nonphysical states. By means of the Lamé equation, we clarify the nature of these peculiar states, and show that they encode essential information not only on the original hyperbolic Pöschl–Teller system, but also on its singular hyperbolic and trigonometric modifications, and reflect the intimate relation of the model to a free-particle system.

PACS number: 11.30.Na

1. Introduction

Both on the classical and quantum levels, symmetries are behind the special properties of the systems. Sometimes symmetries appear in a hidden form like it happens, for instance, in the case of a spontaneously broken symmetry. Another, well-known mechanical example is provided by the model of hydrogen atom being the quantum analog of the Kepler problem, in which a hidden symmetry associated with the Laplace–Runge–Lenz vector underlies a specific degeneration of the spectrum [1]. Recently, it was found [2, 3] that some well-studied quantum mechanical systems exhibit a bosonized supersymmetry [4] in a hidden form. The hidden supersymmetry manifests explicitly the main characteristics of the systems and may have a linear, or nonlinear [5] character. Hidden supersymmetry of a linear form appears in the bound state Aharonov–Bohm effect and the Dirac delta potential problems [2]. The pure bosonic quantum Pöschl–Teller (PT) [2] and Lamé [3] systems display hidden supersymmetry of a nonlinear form.

The PT and Dirac delta quantum problems are special limits of the Lamé equation. In the form of the periodic quantum problem, the latter system underlies diverse models and mechanisms in field theory [6, 7], nonlinear wave physics [8], cosmology [9], condensed matter physics [10–12] and statistical mechanics [13]. A Jacobian form of Lamé equation [14, 15] usually used in physics is

$$H_j^{\mathrm{L}}\Psi = 0, \qquad H_j^{\mathrm{L}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + j(j+1)k^2\,\mathrm{sn}^2(x,k) + c, \tag{1.1}$$

where $\mathrm{sn}(x,k) \equiv \mathrm{sn}\, x$ is the Jacobi elliptic sine function, k, $0 < k < 1$, is the modular elliptic parameter, while j and $c = c(j,k)$ are real constants. Equation (1.1) can be treated as a Schrödinger one-dimensional equation with a doubly periodic potential, in which $-c$ has a sense of an eigenenergy.

When the modular parameter takes its limiting values, we obtain two different systems. For $k = 0$ (and finite j), the potential term disappears from H_j^{L} and the Hamiltonian corresponds to a free particle. Meanwhile in the limit $k = 1$, the real period of the potential turns into infinity, and (1.1) is reduced to the Pöschl–Teller system, $H_j^{\mathrm{L}} \rightarrow H_j^{\mathrm{PT}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - j(j+1)\,\mathrm{sech}^2 x + c'$. Since the periods of the elliptic function $\mathrm{sn}^2\, x$ are $2\mathbf{K}$ and $2\mathrm{i}\mathbf{K}'$, while $\mathrm{sech}^2 x$ has the imaginary period $\mathrm{i}\pi$, the potential of the Lamé system can be treated as a certain periodic superposition of the PT potentials [16],

$$H_j^{\mathrm{L}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - j(j+1)\left(\frac{\pi}{2\mathbf{K}'}\right)^2 \sum_{l=-\infty}^{\infty} \mathrm{sech}^2\left(\frac{\pi}{2\mathbf{K}'}[x - 2l\,\mathbf{K}]\right) + j(j+1)\frac{\mathbf{E}'}{\mathbf{K}'} + c. \tag{1.2}$$

What makes the Lamé and Pöschl–Teller models to be particularly interesting are those remarkable properties appearing when the parameter j takes integer values n (for the sake of definiteness we assume $n = 1, 2, \ldots$). For these special values the Lamé equation describes a *finite-gap* quantum periodic system, while the PT system is *reflectionless*.

Behind these special properties, there emerges the hidden nonlinear supersymmetry. For $j = n$, both systems have nontrivial integrals of motion in the form of differential operators of order $2n + 1$. These corresponding odd integrals of motion, Q_n^{L} and Q_n^{PT}, together with reflection (being an obvious symmetry) play the role of the supercharge and grading operators of the hidden nonlinear supersymmetry [2, 3].

Though the nonlinear supersymmetry in both models has a somewhat similar structure, there are essential differences between its realizations. In both systems all the physical singlet states are annihilated by the supercharges, but in the PT, unlike the Lamé case, the supercharge has also non-normalizable, nonphysical (formal) zero modes. This difference is reflected in the nonlinear superalgebraic structure. The square of the supercharge in both systems gives a polynomial of order $2n + 1$ in a corresponding Hamiltonian operator. For (1.1), we get the spectral polynomial with all the roots to be simple and equal to the energies of the edges of the allowed bands. However, for the PT system, the polynomial has n double roots associated with the bound states, while one simple root corresponds to the lowest, singlet state of the continuous spectrum. Having in mind these nonlinear superalgebraic relations between corresponding supercharges and Hamiltonians, in Lamé (L) and PT systems with $j = n$ we deal, respectively, with nondegenerate and degenerate hyperelliptic curves of genus $g = n$,

$$L: \quad y^2 = \prod_{k=0}^{2n+1}(z - z_k), \qquad PT: \quad y^2 = (z - z_{2n+1})\prod_{k=0}^{n}(z - z_k)^2, \qquad z_k \neq z_{k'}. \tag{1.3}$$

The existence of these differences gives rise to the following questions: what is the relation between the two systems from the point of view of the hidden supersymmetry, and, on the other hand, if the supercharges are fundamentals objects that contain all the information on the systems, what is the origin and nature of nonphysical states from the PT supercharge kernel? It is the purpose of the present paper to answer these questions.

The paper is organized as follows. In section 2 the hidden nonlinear supersymmetric structure of the PT and Lamé systems is reviewed, and the structure of their supercharge kernels and relation between them is discussed. In section 3 the nature of non-normalizable, nonphysical states of the PT supercharge kernel is clarified, and their origin in the light of the Lamé equation is investigated. We conclude in section 4 with a brief summary.

2. Hidden supersymmetry of the Lamé and PT systems

A part of the spectrum of the Lamé and PT systems with $j = n$ is doubly degenerated. For (1.1) this corresponds to the energies of the quasiperiodic (Bloch–Floquet) states of the internal part of the valence and conduction bands, while in the PT system these are the energies of the scattering states (except the lowest one). Double degeneration of the energy levels is a characteristic feature of the $N = 2$ supersymmetry generated by two supercharges. On the other hand, there are $2n + 1$ singlet states corresponding to the edges of the allowed bands in the Lamé system, and $n + 1$ singlets corresponding to n bound states plus one nondegenerate lowest state of the scattering sector in the PT system. In both cases, the number of singlet states is greater than 1, that is typical for the nonlinear supersymmetry [5, 17].

The nonlinear supersymmetry in both systems is generated by the local, $Q_n^{\mathrm{L,PT}} = Q_n$, and nonlocal (due to a nonlocal nature of R), $\tilde{Q}_n = \mathrm{i}RQ_n$, supercharges

$$[Q_n, H_n] = [\tilde{Q}_n, H_n] = 0, \qquad \{Q_n, \tilde{Q}_n\} = 0, \tag{2.1}$$

$$Q_n^2 = \tilde{Q}_n^2 = P_{2n+1}(H_n), \tag{2.2}$$

where $P_{2n+1}(H_n)$ is a polynomial of order $2n + 1$ in Hamiltonian $H_n = H_n^{\mathrm{L,PT}}$ and R is the reflection operator $R\Psi(x) = \Psi(-x)$ identified as the grading operator, $[R, H_n] = 0, \{R, Q_n\} = \{R, \tilde{Q}_n\} = 0, R^2 = 1$. From the structure of the nonlocal supercharge $\tilde{Q}_n$ it is clear that its kernel coincides with that of the local supercharge Q_n.

The $2n + 1$ nondegenerate states of the edges of the allowed bands of system (1.1) are given by the Lamé polynomials [14, 15] of the form

$$\mathrm{sn}^r x\, \mathrm{cn}^s x\, \mathrm{dn}^t x\, F_p(\mathrm{sn}^2 x), \tag{2.3}$$

where $F_p(\mathrm{sn}^2 x)$ is a polynomial of order p in $\mathrm{sn}^2 x, r, s, t = 0$ or 1 and $r + s + t + 2p = n$. The states (2.3) are annihilated by Q_n^{L}, and thus form the supercharge kernel $K_n^{\mathrm{L}} = \ker Q_n^{\mathrm{L}}$.

For $n = 0$, the Lamé system reduces to a trivial case of a free particle, and the momentum operator $Q_0^{\mathrm{L}} = -\mathrm{i}D, D = \frac{\mathrm{d}}{\mathrm{d}x}$, is identified as a supercharge. The first nontrivial case corresponding to $n = 1$ is characterized by the supercharge

$$\mathrm{i}Q_1^{\mathrm{L}} = D^3 + fD + \tfrac{1}{2}f', \tag{2.4}$$

where $f' = \frac{\mathrm{d}}{\mathrm{d}x}f$ and f is a doubly periodic function being, up to a shift of the argument, the Weiesstrass elliptic $\wp$-function with periods $\omega_1 = 2\mathbf{K}$ and $\omega_2 = 2\mathrm{i}\mathbf{K}'$,

$$f := 1 + k^2 - 3k^2\,\mathrm{sn}^2 x = -3\wp(x + \mathrm{i}\mathbf{K}'). \tag{2.5}$$

The supercharge for an arbitrary even (odd) n is constructed recursively via $Q_0^{\mathrm{L}}\ \big(Q_1^{\mathrm{L}}\big)$,

$$Q_n^{\mathrm{L}} = \Lambda_n Q_{n-2}^{\mathrm{L}}, \qquad n > 1, \tag{2.6}$$

where Λ_n is a differential operator of order 4,

$$\begin{aligned}\Lambda_n = D^4 &+ [2n(n-1)+1]fD^2 + \left[\tfrac{4}{3}(n-1)\big(n-\tfrac{1}{2}\big)\big(n+\tfrac{3}{2}\big) + \tfrac{1}{2}\right]f'D \\ &+ (n-1)^2\left[\tfrac{2}{3}\Big(\big(n+\tfrac{1}{2}\big)^2 + \tfrac{1}{2}\Big)f'' + n^2 f^2\right].\end{aligned} \tag{2.7}$$

Though the Lamé polynomials (2.3) as well as the corresponding energies $E^{\mathrm{L}}_{n,l}$ of the band edges can be found in an analytic form for $n \leqslant 8$ [18], in correspondence with the recurrent structure of the supercharges, their kernels can be presented explicitly in general case in terms of monomials in $\mathrm{sn}\, x$, $\mathrm{cn}\, x$ and $\mathrm{dn}\, x$ [3],

$$K^{\mathrm{L}}_n = \left\{K^{\mathrm{L}}_{n-2}, \mathrm{dn}^n\, x, \mathrm{cn}\, x\, \mathrm{dn}^{n-1}\, x, \mathrm{sn}\, x\, \mathrm{dn}^{n-1} x, \mathrm{cn}\, x\, \mathrm{sn}\, x\, \mathrm{dn}^{n-2} x\right\}, \qquad n > 1, \tag{2.8}$$

where $K^{\mathrm{L}}_0 = 1$, $K^{\mathrm{L}}_1 = \{\mathrm{dn}\, x, \mathrm{cn}\, x, \mathrm{sn}\, x\}$. The monomial elliptic functions of the kernel are certain linear combinations of Lamé polynomials.

The square of the supercharge (2.2) is given by the Lamé spectral polynomial

$$P^{\mathrm{L}}_{2n+1}\left(H^{\mathrm{L}}_n\right) = \prod_{l=0}^{2n}\left(H^{\mathrm{L}}_n - E^{\mathrm{L}}_{n,l}\right), \tag{2.9}$$

where $E^{\mathrm{L}}_{n,l}$, $l = 0, \ldots, 2n$, are the eigenvalues of the edges of the allowed bands.

The limit $k = 1$ preserves the hidden supersymmetry and transforms the periodic finite-gap Lamé equation (1.1) into a reflectionless Pöschl–Teller system

$$H^{\mathrm{L}}_n \xrightarrow[k=1]{} H^{\mathrm{PT}}_n = -D^2 - n(n+1)\mathrm{sech}^2 x + n^2 = D^{\dagger}_n D_n, \tag{2.10}$$

where $D_n = -D^{\dagger}_{-n} = \frac{\mathrm{d}}{\mathrm{d}x} + n \tanh x$. The states

$$\psi_{n,l} = \mathcal{D}^l_{-n} \cosh^{l-n} x, \qquad l = 0, 1, \ldots, n, \tag{2.11}$$

with

$$\mathcal{D}^0_{-n} = 1, \qquad \mathcal{D}^1_{-n} = \mathcal{D}_{-n},$$
$$\mathcal{D}^l_{-n} = D_{-n} D_{-n+1} \ldots D_{-n+l-1}, \qquad l = 2, \ldots, n, \tag{2.12}$$

represent n bound states corresponding to $l = 0, \ldots, n-1$, while the state with $l = n$ is the lowest state from the continuous part of the spectrum. Their energies are given by

$$E^{\mathrm{PT}}_{n,l} = n^2 - (n-l)^2, \qquad l = 0, \ldots, n. \tag{2.13}$$

A specific choice of the constant shift in (2.10) corresponds to the zero energy value of the ground state $\psi_{n,0}$. These $n+1$ singlet eigenstates of the Hamiltonian constitute a part of the kernel $K^{\mathrm{PT}}_n = \ker Q^{\mathrm{PT}}_n$ of the supercharge. The latter can be obtained directly from (2.6) by taking the limit $k = 1$. This supercharge can be represented in a more elegant form [2],

$$Q^{\mathrm{PT}}_n = \mathrm{i} D_{-n} D_{-n+1} \ldots D_n. \tag{2.14}$$

Its square gives a corresponding polynomial (2.2),

$$P^{\mathrm{PT}}_{2n+1}\left(H^{\mathrm{PT}}_n\right) = \left(H^{\mathrm{PT}}_n - E^{\mathrm{PT}}_{n,n}\right)\prod_{l=0}^{n-1}\left(H^{\mathrm{PT}}_n - E^{\mathrm{PT}}_{n,l}\right)^2. \tag{2.15}$$

This polynomial is a $k = 1$ limit of the Lamé spectral polynomial (2.9), but unlike the latter, it has n double roots corresponding to the singlet bound states energies, while its one simple root corresponds to the energy of the singlet lowest state from the continuous spectrum. All the associated $n+1$ singlet eigenstates (2.11) are annihilated by the supercharge. However, supercharge (2.14) is the differential operator of the order $2n+1$, and its complete kernel (without taking into account the question of normalizability of the states) of dimension $2n+1$ is given recursively as

$$K^{\mathrm{PT}}_n = \left\{K^{\mathrm{PT}}_{n-2}, \cosh^{-n} x, \cosh^{-n} x \sinh x, \cosh^{n-2} x \sinh x, \cosh^n x\right\}, \qquad n > 1, \tag{2.16}$$

where $K^{\mathrm{PT}}_0 = 1$, $K^{\mathrm{PT}}_1 = \{\mathrm{sech}\, x, \tanh x, \cosh x\}$. It is spanned by the states

$$K^{\mathrm{PT}}_n = \{\cosh^s x \sinh^r x, s = -n, -n+2, \ldots, n-2r, r = 0, 1\}, \tag{2.17}$$

and can be rearranged as follows:

$$K_n^{\mathrm{PT}} = \{\hat{K}_n^{\mathrm{PT}}, \kappa_n^{\mathrm{PT}}, \tilde{K}_n^{\mathrm{PT}}\}, \qquad \dim \hat{K}_n^{\mathrm{PT}} = \dim \tilde{K}_n^{\mathrm{PT}} = n, \qquad \dim \kappa_n^{\mathrm{PT}} = 1. \tag{2.18}$$

Here $\hat{K}_n^{\mathrm{PT}}$ corresponds to the normalizable (with respect to the ordinary scalar product on $\mathbb{R}^1$) functions

$$\hat{K}_n^{\mathrm{PT}} = \{\cosh^s x \sinh^r x, r = 0, 1, \quad \begin{Bmatrix} s = -n, -n+2, \ldots, -2, & n = 2m > 0 \\ s = -n, -n+2, \ldots, -(2r+1), & n = 2m+1 \end{Bmatrix}, \tag{2.19}$$

being linear combinations of the n bound states, while

$$\kappa_n^{\mathrm{PT}} = \begin{Bmatrix} 1, & n = 2m, \\ \tanh x, & n = 2m+1 \end{Bmatrix} \tag{2.20}$$

are linear combinations of the singlet scattering state $\psi_{n,n}$ and bound states $\psi_{n,l}, l = 0, \ldots, n-1$. These are the $n+1$ states corresponding to the $k = 1$ limit of the supercharge kernel (2.8) of the Lamé system, $K_n^{\mathrm{L}} \underset{k=1}{\longrightarrow} \{\hat{K}_n^{\mathrm{PT}}, \kappa_n^{\mathrm{PT}}\}$. In this limit, the period of the Lamé equation tends to infinity, the valence bands shrink, and two edge states (and their energies) of the same band converge smoothly in one bound state (and corresponding energy) of the Pöschl–Teller system. The states of the continuous band of (1.1) in this limit are transformed into the states of the continuous spectrum of (2.10), and the singlet edge state of the conduction band is transformed into the first (lowest) singlet state of the continuous spectrum. From another point of view, since in the limit $k = 1$ Jacobi functions $\mathrm{cn}\, x$ and $\mathrm{dn}\, x$ are reduced to the same function $\mathrm{sech}\, x$, two different Lamé polynomials are transformed into the same function in terms of associated Legendre functions of the variable $\tanh x$ [15].

Therefore, from the point of view of the $k = 1$ limit of the Lamé system, the origin of the non-normalizable nonphysical states $\cosh^s x, s \geqslant 1$, and $\cosh^{s'} x \sinh x, s' \geqslant 0$, from the n-dimensional subspace $\tilde{K}_n^{\mathrm{PT}}$ of the total kernel K_n^{PT} of the Pöschl–Teller supercharges seems to be mysterious.

3. The nature and origin of $\tilde{K}_n^{\mathrm{PT}}$

Before investigating the question on the origin of the non-normalizable nonphysical states of the supercharge kernel from the point of view of the associated Lamé system, we clarify their nature within the framework of the Pöschl–Teller system itself.

First we note that the states of the complete kernel (2.17) with the same parity but different values of the parameter s can be related by the Hamiltonian operator

$$(H_n - E_{n,n-|s+r|}) \cosh^s x \sinh^r x = C_{s,n} \cosh^{s-2} x \sinh^r x, \tag{3.1}$$

where $s = -n, -n+2, \ldots, n-2r$, $C_{s,n} = s(s-1) - n(n+1)$ and $E_{n,l} = E_{n,l}^{\mathrm{PT}}$ are the energies given by equation (2.13). On the other hand, all the physical singlet states (2.11) (both from the bound and continuous parts of the spectrum) can be produced by the action of the polynomial in Hamiltonian operator on the states of the supercharge kernel belonging to $\hat{K}_n^{\mathrm{PT}}$ or κ_n^{PT},

$$\prod_{s=0}^{\frac{1}{2}(n-l)-1} (H_n - E_{n,2s+r}) \cosh^{-l} x \sinh^r x = \psi_{n,n-l+r}(x), \tag{3.2}$$

where $l = n-2, n-4, \ldots, 2r$ for even n, and $l = n-2, n-4, \ldots, 1$ for odd n, with $r = 0, 1$.

Combining relations (3.1) and (3.2) we conclude that any physical singlet state can be obtained from any nonphysical (exponentially increasing) state (2.17) of the supercharge kernel by applying to it a certain polynomial operator in the Hamiltonian.

The states of the complete supercharge kernel (2.17) have also the following property. Let us identify their logarithmic derivatives

$$W_{0,s} = -\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cosh^s x) = -s\tanh x, \qquad W_{1,s} = -\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cosh^s \sinh x) = W_{0,s} - \coth x, \tag{3.3}$$

as superpotentials in the sense of a usual linear supersymmetry and construct the corresponding superpartner Hamiltonians $H^{\pm}_{r,s} = -D^2 + W^2_{r,s} \pm W'_{r,s}$, where we assume that $s \in \mathbb{Z}$. Then we get

$$H^+_{0,s} = -D^2 - s(s+1)\,\mathrm{sech}^2 x + s^2, \qquad H^-_{0,s} = H^+_{0,s-1} + 2s - 1, \tag{3.4}$$

$$H^+_{1,s} = -D^2 - s(s+1)\,\mathrm{sech}^2 x + (s+1)^2 + 2\,\mathrm{cosech}^2 x, \qquad H^-_{1,s} = H^+_{0,s-1} + 4s. \tag{3.5}$$

Since $H^+_{0,s} = H^-_{0,-s}$, from the point of view of such a construction both physical and nonphysical states of the complete kernel of the supercharge are, in fact, equivalent, and every time we produce either a (shifted) reflectionless Pöschl–Teller system with the corresponding value of the constant parameter, or a free particle $\left(H^+_{0,0}, H^-_{0,1}, H^-_{1,1}\right)$, or the generalized Pöschl–Teller system $\left(H^+_{1,s}\right)$ [19]. In particular, note that the states $\cosh^{\pm n} x$ generate exactly the 'parent' system given by $H^+_{0,n} = H^-_{0,-n} = H^{\mathrm{PT}}_n$. From relations (3.4) it also follows the well-known fact lying behind the reflectionless property: the supersymmetric partner of the PT system with $n = 1$ is a free particle, while the PT system with $j = n > 1$ is related to a free particle via usual, linear supersymmetry in n steps [19].

Taking into account the relations $H^+_{0,s-1}\cosh^s x = (1 - 2s)\cosh^s x$, $H^+_{0,s-1}(\cosh^s x \sinh x) = -4\cosh^s \sinh x$, and $H^+_{0,-s} = H^+_{0,s-1} + 2s - 1$, we get

$$H^{\mathrm{PT}}_n \cosh^{-n} x = 0, \qquad H^{\mathrm{PT}}_n(\cosh^{-n}\sinh x) = (2n-1)\cosh^{-n} x \sinh x, \tag{3.6}$$

$$H^{\mathrm{PT}}_{n-1}\cosh^n x = -(2n-1)\cosh^n x, \qquad H^{\mathrm{PT}}_{n-1}(\cosh^n x \sinh x) = -4n\cosh^n x \sinh x. \tag{3.7}$$

While two physical states from (2.17) with $s = -n, r = 0, 1$ are the first (ground) and the second singlet eigenstates of the system H^{PT}_n, other physical states of the kernel (2.17) are the first and the second singlet eigenstates of the PT systems H^{PT}_s with corresponding values of the parameter $0 < s < n$. Moreover, according to equation (3.7), the *nonphysical* states of the kernel K^{PT}_n can also be identified as *non-normalizable* eigenstates of the Pöschl–Teller (or shifted free particle) systems with corresponding *negative* eigenvalues.

Let us apply a Wick rotation to the PT potential to see further evidence for the importance of the nonphysical states of the PT supercharge kernel. The rotation can be realized by 'restoring' a frequency parameter ω in the potential term, $H^{\mathrm{PT}}_n = -D^2 - n(n+1)\omega^2\,\mathrm{sech}^2\,\omega x + n^2\omega^2$, subsequent substitution $\omega \to \mathrm{i}\omega$, and then putting again $\omega = 1$. In this way, we get the trigonometric Pöschl–Teller system, $H^{\mathrm{PT}}_n \to \tilde{H}^{\mathrm{PT}}_n = -D^2 + n(n+1)\sec^2 x - n^2$. Under such a procedure, nonphysical states $\cosh^n x$ and $\cosh^n x \sinh x$ are transformed into the trigonometric counterparts $\cos^n x$ and $\cos^n x \sin x$ belonging to the kernel of the transformed supercharge operator. But now they are physical states of the trigonometric PT system with $j = n - 1$, which, in correspondence with equation (3.7), are the two first (lower) eigenstates of the $\tilde{H}^{\mathrm{PT}}_{n-1}$. Analogously, other nonphysical states of the supercharge kernel of the hyperbolic PT system are transformed into physical states of the supercharge kernel of the trigonometric

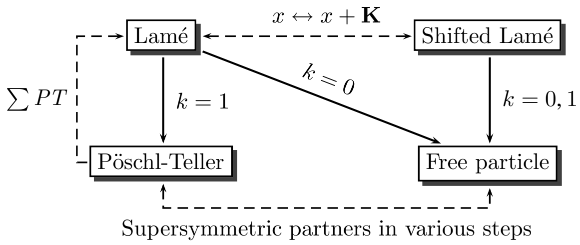

Figure 1. The relationship between PT, Lamé and free particle systems.

PT system. On the other hand, normalizable states from the physical part of the supercharge kernel are transformed into nonphysical states (which violate necessary boundary conditions at $x = \pm\pi/2$) of the trigonometric PT supercharge kernel.

Let us return to the Lamé system. As we have seen, the limit $k = 1$ produces from the Lamé supercharge kernel only the physical states of the Pöschl–Teller supercharge kernel. We shall show that nonphysical states of the PT supercharge kernel can also be obtained in the same limit proceeding from the Lamé system. For this we note that due to periodicity, a constant real shift of the argument in the Lamé Hamiltonian results just in a shift of the potential along a real line, but does not change the spectrum of the system and its special properties. With this observation, let us shift the argument for a half of the real period of the potential [20], $x \to x + \mathbf{K}$,

$$H_n^{\mathrm{L}} \underset{x=x+\mathbf{K}}{\longrightarrow} H_n^{L+\mathbf{K}} = -D^2 + n(n+1)[1 - k'^2\,\mathrm{dn}^{-2}(x,k)] + c. \tag{3.8}$$

Shifting the argument in the corresponding formulas for the Lamé system, we get the supercharge $Q_n^{L+\mathbf{K}}$ for system (3.8) and its corresponding kernel $K_n^{L+\mathbf{K}}$. The difference of the shifted system (3.8) in comparison with the original one (1.1) is that in both limits $k = 0$ and $k = 1$ it transforms into the free particle but with different additive constants. The relationship between PT, Lamé and free particle systems is summarized in figure 1.

In the limit $k = 1$, Hamiltonian (3.8) and the supercharge $Q_n^{L+\mathbf{K}}$ are transformed into

$$H_n^{\mathrm{free}} = -D^2 + n^2 \tag{3.9}$$

$$Q_n^{\mathrm{free}} = -\mathrm{i}D(D^2 - 1^2)\cdots(D^2 - (n-1)^2)(D^2 - n^2). \tag{3.10}$$

Operator (3.10) is obviously an integral of motion, which is reduced to a polynomial of order n in Hamiltonian (3.9) multiplied by D. Its kernel is spanned by the functions $\cosh sx$, $\sinh sx$, $s = 0, 1, \ldots, n$. In another form the kernel can be obtained directly from (2.8),

$$K_n^{\mathrm{L}} \underset{x=x+\mathbf{K}}{\longrightarrow} K_n^{L+\mathbf{K}} \underset{k=1}{\longrightarrow} K_n^{\mathrm{free}} = \left\{K_{n-1}^{\mathrm{free}}, \cosh^n x, \cosh^{n-1}\sinh x\right\}, \qquad n > 0, \quad K_0^{\mathrm{free}} = 1. \tag{3.11}$$

The functions which belong to (3.11) are some linear combinations of non-normalizable eigenstates of the Hamiltonian (3.9). The special feature of (3.11) is that it is composed by functions from (2.16), in particular, by functions from $\tilde{K}_n^{\mathrm{PT}}$. Besides, (3.11) contains the functions from $\tilde{K}_{n+1}^{\mathrm{PT}}$. Supercharge (3.10) also annihilates a constant, and the structure of its kernel (3.11) can be summarized as follows:

$$K_n^{\mathrm{free}} = \left\{\tilde{K}_n^{\mathrm{PT}}, \kappa_{2n}^{\mathrm{PT}}, \{n\text{functions }\} \in \tilde{K}_{n+1}^{\mathrm{PT}}\right\}. \tag{3.12}$$

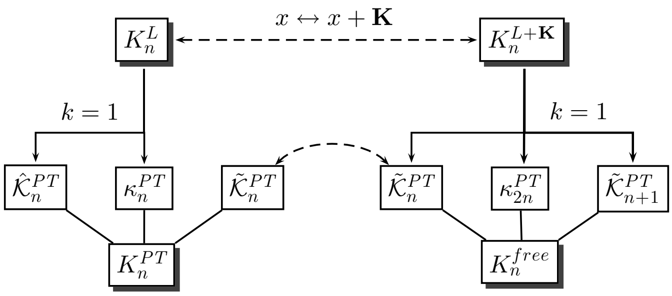

Figure 2. Supercharge kernels of $k = 1$ Lamé, shifted Lamé, PT and free particle systems.

The relationship between supercharge kernels of $k = 1$ Lamé, shifted Lamé, PT and free particle systems is presented in figure 2.

As an example, let us consider the simplest case $n = 1$, for which the Hamiltonians (1.1) and (3.8) are

$$H_1^{\mathrm{L}} = -D^2 + 2k^2\,\mathrm{sn}^2\,x - k^2, \qquad H_1^{L+\mathbf{K}} = -D^2 - 2k'^2\,\mathrm{dn}^{-2}\,x + 2 - k^2. \tag{3.13}$$

The systems have two allowed bands, and so, three energy eigenstates associated with band edges. These states and corresponding eigenvalues are summarized in the table below:

	Lamé	Shifted Lamé	$E_{1,l}$
$\Psi_{1,0}$	$\mathrm{dn}\,x$	$1/\mathrm{dn}\,x$	0
$\Psi_{1,1}$	$\mathrm{cn}\,x$	$\mathrm{sn}\,x/\mathrm{dn}\,x$	$1-k^2$
$\Psi_{1,2}$	$\mathrm{sn}\,x$	$\mathrm{cn}x/\mathrm{dn}\,x$	1

(3.14)

The edge band states are zero modes of the corresponding supercharges Q_1^{L} and $Q_1^{L+\mathbf{K}}$, where the latter is obtained from (2.4), (2.5) using the relation $\mathrm{sn}(x+\mathbf{K}) = \mathrm{cn}\,x/\mathrm{dn}\,x$. In the limit $k = 1$, the Hamiltonian H_1^{L} is transformed into $H_1^{\mathrm{PT}} = -D^2 - 2\,\mathrm{sech}^2\,x + 1$. Its unique bound state $\psi_{1,0} = \mathrm{sech}\,x$ originates from the states $\Psi_{1,0}$ and $\Psi_{1,1}$ of the edges of the contracting valence band. The singlet state of the continuous spectrum, $\psi_{1,1} = \tanh x$, originates from the state of the edge of the conductance band, $\Psi_{1,2}^{\mathrm{L}} = \mathrm{sn}\,x$. Together with the non-normalizable nonphysical state $\cosh x$, they form the kernel of Q_1^{PT}. The Hamiltonian of the shifted Lamé system is transformed in this limit into $H_1^{\mathrm{free}} = -D^2 + 1$, and its edge band states (3.14) are transformed into non-normalizable states of system (3.9) with $n = 1$. Only in the case $n = 1$ systems (1.1) and (3.8) form a pair of supersymmetric partners [20]. In correspondence with this, in the limit $k = 1$ the PT system is a superpartner of a free particle. At $k = 1$, the supercharge $Q_1^{\mathrm{L}+\mathbf{K}}$ is transformed into $Q_1^{\mathrm{free}} = -\mathrm{i}(D-1)D(D+1) = iH_1^{\mathrm{free}}D$, and the physical states which form the kernel (3.11) are

$$K_1^{\mathrm{free}} = \{\cosh x, 1, \sinh x\} = \left\{\tilde{K}_1^{\mathrm{PT}}, \kappa_2^{\mathrm{PT}}, \sinh x \in \tilde{K}_2^{\mathrm{PT}}\right\}. \tag{3.15}$$

The two physical states spanning the subspace $\left\{\hat{K}_1^{\mathrm{PT}}, \kappa_1^{\mathrm{PT}}\right\}$ of the supercharge kernel K_1^{PT} originate from the supercharge kernel of system (1.1). The missing non-normalizable nonphysical zero mode $\cosh x \in \tilde{K}_1^{\mathrm{PT}}$ is provided by the shifted Lamé system in the limit $k = 1$. But the kernel K_1^{free} contains two more zero modes, which are also nonphysical states and which are related to the system H_2^{PT}. In particular, $\sinh x \in \tilde{K}_2^{\mathrm{PT}}$ and a constant function from K_1^{free} corresponds to κ_2^{PT}. Since $\dim \tilde{K}_2^{\mathrm{PT}} = 2$, there is a missing state to complete the kernel $\tilde{K}_2^{\mathrm{PT}}$. It is provided by K_2^{free}, and in this way one can successively continue.

We have analysed the Lamé equation by shifting its argument in the half of the real period. But it seems to be natural also to look what happens under shifting the argument for the half of the imaginary period, $x \to x + \mathrm{i}\mathbf{K}'$, as well as under the 'diagonal' shift

$$x \to x + \mathbf{K} + \mathrm{i}\mathbf{K}', \tag{3.16}$$

with taking subsequently the limits $k = 0$ and $k = 1$. The results are summarized in the following table:

	$V_n^{\mathrm{L}}(x)$	$V_n^{\mathrm{L}}(x+\mathbf{K})$	$V_n^{\mathrm{L}}(x+\mathrm{i}\mathbf{K}')$	$V_n^{\mathrm{L}}(x+\mathbf{K}+\mathrm{i}\mathbf{K}')$
k	$n(n+1)k^2\,\mathrm{sn}^2\,x$	$-n(n+1)k'^2\,\mathrm{dn}^{-2}(x,k)$	$\frac{n(n+1)}{\mathrm{sn}^2\,x}$	$n(n+1)\frac{\mathrm{dn}^2 x}{\mathrm{cn}^2 x}$
$k=1$	$-n(n+1)\,\mathrm{sech}^2 x$	b_2	$\frac{n(n+1)}{\sinh^2 x}$	b_4
$k=0$	b_1	b_3	$\frac{n(n+1)}{\sin^2 x}$	$\frac{n(n+1)}{\cos^2 x}$

where b_i, $i = 1, \ldots, 4$, are some constants. These potentials correspond to the Pöschl–Teller, free particle (b_i), or Pöschl–Teller-related systems. Potentials of Pöschl–Teller-related systems have singularities appearing from the poles of elliptic functions displaced to the real line. For all the systems the supercharges of the hidden supersymmetry are obtained from the supercharge (2.6) of the original Lamé system. For the systems with singular potentials supersymmetry is of a fictitious nature [21]. In such systems, the resulting supercharges commute with corresponding Hamiltonians, but acting on the physical states they produce non-normalizable singular states which violate the boundary conditions, and so, are not physical states.

4. Conclusion

We have clarified the nature and the origin of the kernel of the supercharge of the hidden nonlinear supersymmetry of the Pöschl–Teller system by investigating the $k = 0$ and $k = 1$ limits of the associated periodic Lamé, or appropriately translated Lamé equation. We have showed that in spite of the nonphysical nature of the non-normalizable states, which constitute a part of the kernel, they encode essential information on the original hyperbolic PT system, and its singular hyperbolic and trigonometric modifications, and reflect the intimate relation of the system to a free particle. In particular, it is interesting to note that under the Wick rotation, which transforms hyperbolic PT into its singular trigonometric counterpart, and corresponds to a diagonal translation (3.16) of the Lamé system for a half of the complex period, the nature of physical and nonphysical states of the PT supercharge kernel is interchanged. This effect is related to the duality in the Lamé model discussed by Dunne and Shifman [22].

To conclude, having in mind that the nonlinear superalgebraic structures, given by equations (2.2), (2.9) and (2.15), have a form of nondegenerate and degenerate hyperelliptic curves (1.3), it would also be very interesting to understand the essential differences between the hidden supersymmetry (as well as its origin) in the Lamé and PT systems within the differential geometric framework of genus n Riemann surfaces[1].

Acknowledgments

The work has been partially supported by CONICYT and by FONDECYT under grant 1050001.

[1] In a different, but somewhat related, context, see also [22].

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